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$\mathcal{N} = 4$ super Yang-Mills matrix integrals for almost all simple gauge groups

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Abstract: In this paper the partition function of $\mathcal{N} = 4$ $D = 0$ super Yang-Mills matrix theory with arbitrary simple gauge group is discussed. We explicitly computed its value for all classical groups of rank $r \leqslant 11$ and for the exceptional groups G_2, F_4 and E_6. In the case of classical groups of arbitrary rank we conjecture general formulas for the B_r, C_r and D_r series in addition to the known result for the A_r series. Also, the relevant boundary term contributing to the Witten index of the corresponding supersymmetric quantum mechanics has been explicitly computed as a simple function of rank for the orthogonal and symplectic groups SO$(2N+1)$, Sp$(2N)$, SO$(2N)$.

Keywords: Field Theories in Lower Dimensions, M(atrix) Theories, Supersymmetric Effective Theories.

Contents

1. Introduction

The super Yang-Mills matrix models are obtained by the dimensional reduction to 0 dimensions of the supersymmetric Yang-Mills gauge theories (SYM) in several dimensions. They have already appeared in some contexts. Firstly, the action of $\mathrm{SU}(N)$ model describes in the leading order the world volume potential on a stack of N $D(-1)$-branes [1]. Secondly, these models (in the limit $N \to \infty$) are relevant for the constructions [2] which are believed to provide a non-perturbative formulation of superstring theory. Since the dynamics of the reduced theory can capture some features of dynamics of the full unreduced theory, it is not surprising to encounter them in multi-instanton calculus [3]. Finally, they are closely connected with the question about the number of normalizable ground states in the SYM quantum mechanics [4]–[9].

In this paper we consider the partition function I_b of $\mathcal{N}=4$ supersymmetric Yang-Mills matrix model obtained by the dimensional reduction to 0 dimensions from $D=4$ $\mathcal{N}=1$ supersymmetric Yang-Mills gauge theory with arbitrary simple gauge group G

$$I_b(\mathfrak{g}) = \frac{1}{\mathrm{Vol}(G/Z)} \int d\lambda\, dA\, dD e^{-S}\,, \tag{1.1}$$

where the action is

$$S_{YM} = -\,\mathrm{Tr}\left(\frac{1}{4}[A_\mu, A_\nu]^2 + \bar{\lambda}\bar{\sigma}^\mu[A_\mu,\lambda] - 2D^2\right). \tag{1.2}$$

By A_μ, λ and D we denote the dimensional reduction to 0 dimension of four-dimensional vector, 2 component complex Weyl spinor and scalar, respectively. All fields are in the adjoint representation of G: their components belong to the Lie algebra $\mathfrak{g}$ of the group G. The functional integral that defines the dynamics of the theory is reduced to an ordinary finite dimensional integral. This means a tremendous simplification for a computation, nevertheless the reduced theory remains non trivial due to the quartic commutator potential. In this work we will focus on computing the partition function with zero sources (1.1). The measure of integration is obtained from a Killing form on $\mathfrak{g}$. It is unique up to an overall scaling, but the factor $1/\mathrm{Vol}(G/Z)$ (where Z is the center of G) in front of the integral makes the whole expression invariant to rescaling and a topological structure of G. So, the only argument of I_b is the Lie algebra $\mathfrak{g}$.

It should be mentioned, that in the euclidian signature (where we are working) the integral (1.1) is convergent. Indeed, after integrating over λ and D one obtains a homogeneous function of A^a_μ, and then in the spherical coordinate system an integration over the radius $|A|$ can be easily performed. The obtained function of angular coordinates will be singular at some varieties corresponding to the directions where the potential $[A^\mu, A^\nu]^2$ vanishes. Simple counting of powers of the divergence with respect to the corank of these varieties shows that, nevertheless, the integral I_b is convergent [11]. In the simplest case of SU(2) gauge group the integral I_b can be directly computed without any deformation of the integrand [5, 6, 7] (and the result is equal to 1/4). In the case of higher rank simple groups the integral is too sophisticated to be computed directly. In series of works [12] a numerical Monte-Carlo method was applied to approximately calculate I_b and the result has been obtained for all simple Lie groups with rank $\leqslant 3$.

In the paper by G. W. Moore, N. Nekrasov and S. Shatashvili (MNS) [10] the authors managed to reduce the integral $I_b(\mathfrak{su}(N))$ to a contour integral of a rational function using a certain deformation of the integrand and a localization principle that we will briefly review in the section 2. Their method is easily generalized for arbitrary simple Lie group. However, as it will be seen below, the case of $\mathrm{SU}(N)$ group is distinguished by existence of "the determinant formula"[1]

$$\frac{\prod_{1\leqslant i<j\leqslant N}(x_i-x_j)(y_i-y_j)}{\prod_{1\leqslant i,j\leqslant N}(x_i-y_j)} = \det_{ij}\left(\frac{1}{x_i-y_j}\right) \tag{1.3}$$

[1]A bosonic-fermionic correspondence for the correlator of $2N$ primary fermionic fields $\psi(x_i)$ and $\psi^+(y_i)$ in conformal field theory.

which drastically simplifies a computation of the contour integral and eventually allows to obtain the result

$$I_b^{\mathrm{MNS}}(\mathfrak{su}(N)) = \frac{1}{N^2}\,. \tag{1.4}$$

In spite of the fact that the relevant generalization of (1.3) to other Lie groups is not known to us, in this work we will try to compute the contour integrals $I_b^{\mathrm{MNS}}(\mathfrak{g})$ in the case of arbitrary simple Lie algebra $\mathfrak{g}$.

Another approach to the problem of computation I_b exists and it is connected with a notion of the Witten index [13] for a supersymmetric quantum mechanics obtained by the dimensional reduction to one dimension of four dimensional $D = 4$ $\mathcal{N} = 1$ super Yang-Mills theory. The Witten index for quantum mechanics is defined as follows [13]

$$\mathrm{ind}_w = \lim_{\beta\to\infty} \mathrm{Tr}(-1)^F e^{-\beta H} = n_b^0 - n_f^0 \tag{1.5}$$

and it counts the difference between the numbers of normalizable bosonic and fermionic ground states. In supersymmetric quantum mechanics with discrete spectrum ind_w does not depend on β, since bosonic and fermionic states come in pairs for every energy level $E > 0$ and their contribution to ind_w cancel. Therefore, instead of the limit $\beta \to \infty$ one can take $\beta \to 0$ and compute more simple quantity

$$\lim_{\beta\to 0} \mathrm{Tr}(-1)^F e^{-\beta H}\,. \tag{1.6}$$

In terms of path integral[2] the limit $\beta \to 0$ means reduction of $D = 1$ theory to $D = 0$ theory, and the quantity (1.6) is given exactly by the finite dimensional integral I_b that we are interested in.

However, in our case this trick does not work, since the spectrum of the SYM quantum mechanic that we consider is not discrete and $\mathrm{Tr}(-1)^F e^{-\beta H}$ does depend on β. In the case of continuous spectrum the discrete sum over energy levels in (1.5)

$$\sum_k (-1)^{F_k} e^{-\beta F_k} \tag{1.7}$$

is replaced by an integral containing difference between bosonic and fermionic densities,

$$\int_0^\infty dE e^{-\beta E}(n_b(E) - n_f(E)) \tag{1.8}$$

which, generally speaking, is not zero. A convenient way to represent ind_w is the following

$$\mathrm{ind}_w = \lim_{\beta\to\infty} \mathrm{Tr}(-1)^F e^{-\beta H} = I_b - I_d\,, \tag{1.9}$$

$$I_b = \lim_{\beta\to 0} \mathrm{Tr}(-1)^F e^{-\beta H}\,, \tag{1.10}$$

$$I_d = -\int_0^\infty d\beta \frac{d}{d\beta}\, \mathrm{Tr}(-1)^F e^{-\beta H}\,, \tag{1.11}$$

[2]with time $t \in [0 \ldots \beta]$ and periodic boundary conditions with respect to it.

where I_b is known in literature as the bulk contribution ("the principal term") and I_d as the boundary contribution ("the deficit term"). The value of I_d has been rigorously computed only in the case of SU(2) group in [7].

$$I_d(\mathfrak{su}(2)) = \frac{1}{4}\,. \tag{1.12}$$

In [5] it was argued that the only contribution to I_d comes from a region of large A^i where the initial non-abelian SU(2) theory can be approximated by a free effective abelian U(1) theory of particles propagating along "the flat valleys" of the commutator potential $[A^i, A^j]^2$

$$I_d = I_d^{\text{eff}}\,. \tag{1.13}$$

In other words, after gauge fixing all X^i runs mainly along Cartan subalgebra of the gauge group, while fluctuations in all transverse directions are suppressed at large X^i by steep walls of the potential $[X^i, X^j]^2$. Then, the effective theory is a theory of free particles propagating on the moduli space R^{d-1}/Z_2. In this free theory $\text{ind}_w{}^{\text{eff}} = 0$, while principal contribution I_b^{eff} can be easily calculated. In this way the result $I_d = I_d^{\text{eff}} = I_b^{\text{eff}} = 1/4$ was obtained in [5] and it coincided with the rigorous result of [7]. This fact motivated the authors of [8] to suggest that this prescription generalizes for SU(N) and they obtained

$$I_d^{\text{eff}}(\text{SU}(N)) = I_b^{\text{eff}}(\text{SU}(N)) = \left\langle \Psi(-1)^F e^{-\beta H}\mathcal{P}\Psi \right\rangle = \frac{1}{N^2}\,. \tag{1.14}$$

In this equation $\mathcal{P}$ is the projector on the gauge invariant states. The gauge group for this effective free theory is the Weyl group of G and thus $\mathcal{P}$ is equal to the following sum over it

$$\mathcal{P} = \frac{1}{\#W}\sum_{w\in W} M_w\,, \tag{1.15}$$

where M_w represent an action of a Weyl group element $w \in W$ on fields. The obtained result $I_d(\mathfrak{su}(N)) = 1/N^2$ agrees with the result in [10] $I_b^{\text{MNS}}(\text{SU}(N)) = 1/N^2$, provided $\text{ind}_w = 0$. In the paper by V. Kac and A. Smilga [9] the method of [8] was directly generalized for arbitrary simple Lie group and the result was

$$I_d^{\text{KS}}(G) = \frac{1}{\#W(G)} \sideset{}{'}\sum_{w\in W(G)} \frac{1}{\det(1-w)}\,. \tag{1.16}$$

For SU(N) this expression simplifies to $1/N^2$ and agrees with the results of MNS [10] and with direct numerical computation of I_b in [12]. However, in [12] the integral I_b has been calculated for all simple Lie groups of rank $\leqslant 3$ both by numerical (Monte-Carlo) and MNS [10] methods. While numerical and MNS contour integral results coincided with each other, they did not agree with KS formula (1.16). This strongly indicates that the free effective hamiltonian method exploited in [9] fails to calculate I_d.

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2. Localization

We are going to exploit the method used by MNS [10] to compute the integral (1.1) for arbitrary simple Lie group.

Let us briefly review how supersymmetry allows us to greatly simplify an exact analytical calculation of the partition function. An integral like $\int D\varphi e^{-S[\varphi]}$ can be approximately computed using the quasiclassic approximation as the sum $(\det_{ij} \frac{\partial^2 S}{\partial\varphi_i \partial\varphi_j})^{-1/2}$ over the critical points $\partial S/\partial\varphi = 0$ of the action S.[3] It often happens that some symmetry makes all corrections to this approximation vanish. In other words, it means that the integral $\int D\varphi e^{-S[\varphi]}$ is localized on the critical points of the action.[4]

To see how it happens one can deform the integrand by a full derivative in such a way that makes the integral nearly gaussian. Suppose, that S is invariant under the action of some nilpotent derivative[5] operator Q

$$QS = 0\,, \qquad Q^2 = 0\,. \tag{2.1}$$

Integration by parts shows how the partition function changes if the action is deformed by a Q-exact term:

$$Z(g) = \int d\varphi\, e^{-tS(\varphi)+gQW(\varphi)}\,, \tag{2.2}$$

$$\begin{aligned}\partial_g Z &= \int d\varphi\, e^{-tS(\varphi)+gQW(\varphi)} QW(\varphi) \\ &= \int d\varphi\, Q\left(W(\varphi)e^{-tS(\varphi)+gQW(\varphi)}\right) - \int d\varphi\, W(\varphi)Qe^{-tS(\varphi)+gQW(\varphi)}\,. \end{aligned} \tag{2.3}$$

The second term of (2.3) vanishes due to (2.1). The first one can be rewritten as a boundary contribution

$$\int d\varphi \frac{\partial}{\partial\varphi}\left(W(\varphi)e^{-tS(\varphi)+gQW(\varphi)}Q\varphi\right)\,. \tag{2.4}$$

This terms also vanishes if a domain of integration is a compact space without a boundary. In the case of a non-compact space one can neglect a boundary contribution if the integrand of (2.4) decreases exponentially fast when φ tends to the infinity. If an action itself is Q-exact, then we can send the coefficient t in front of the action in the exponent in (2.2) to the infinity, since that does not affect the value of the integral. In such a way it turns into the gaussian one and therefore is equal to the sum over critical points, as claimed above.

Let us apply this method to compute I_b (1.1) [10].

The SYM action

$$S_{YM} = -\operatorname{Tr}\left(\frac{1}{4}[A_\mu, A_\nu]^2 + \bar\lambda\bar\sigma^\mu[A_\mu, \lambda] - 2D^2\right) \tag{2.5}$$

[3] Assuming the measure $\int D\varphi e^{-\varphi^2} = 1$.

[4] A toy example of such a localization is the following integral: $\int_{S^2} d\Omega e^{-\beta\cos\theta} = \frac{2\pi}{\beta}(e^\beta - e^{-\beta})$.

[5] $Q(AB) = (QA)B \pm A(QB)$.

has $\mathcal{N}=4$ $D=0$ supersymmetry transformations which can be obtained by the dimensional reduction from $\mathcal{N}=1$ supersymmetry transformation in $D=4$ dimensions:

$$\delta_\zeta A^\mu = -i\bar{\lambda}\bar{\sigma}^\mu\zeta + i\bar{\zeta}\bar{\sigma}^\mu\lambda\,, \tag{2.6}$$

$$\delta_\zeta\lambda = i\sigma^{\mu,\nu}\zeta[A^\mu, A^\nu] - 2\zeta D\,, \tag{2.7}$$

$$\delta_\zeta D = \frac{1}{2}[A^\mu,\bar{\lambda}]\bar{\sigma}^\mu\zeta + \frac{1}{2}\bar{\zeta}\bar{\sigma}^\mu[A^\mu,\lambda]\,, \tag{2.8}$$

where $\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$.

To rewrite the action and the supersymmetry transformation in a more convenient way we make change of variables as following:

$$\begin{aligned} \lambda_1 &= \eta + i\psi\,, & \lambda_2 &= \chi_1 - i\chi_2\,, \\ u &= \frac{1}{2}\left(A^3 + iA^4\right), & \bar{u} &= \frac{1}{2}\left(A^3 - iA^4\right), \\ X^1 &= A^1\,, & X^2 &= A^2\,, \\ D &= H + \frac{1}{2}\left[A^1, A^2\right]. \end{aligned} \tag{2.9}$$

Then we choose one of four supersymmetry generators δ that acts as following:

$$\begin{aligned} \delta\psi &= H\,, & \delta H &= [u,\psi]\,, \\ \delta\bar{u} &= \eta\,, & \delta\eta &= [u,\bar{u}]\,, \\ \delta X^i &= \chi^i\,, & \delta\chi^i &= [u, X^i]\,, \\ \delta u &= 0\,. \end{aligned} \tag{2.10}$$

It can be easily checked that the action can be rewritten as a δ-exact expression

$$S = \delta\left([u,\bar{u}]\eta + \left[X^i,\bar{u}\right]\chi^i + \left[X^i, X^j\right]\psi\varepsilon_{ij} + H\psi\right). \tag{2.11}$$

The supersymmetry δ squares to a gauge transformation $\delta^2 = [u,\cdot]$ generated by u and thus $\delta^2 = 0$ on gauge invariant quantities.

Therefore, the above described philosophy is suitable in our case: the integrand is localized on classical equations of motion (or a moduli space of the low energy effective theory). However, an additional comprehension arises because this moduli space is not compact and an integration over it is ill defined. In [10] it was suggested to slightly deform the supersymmetry generator $\delta \to \delta_\epsilon$ in such a way that δ_ϵ squares both to the gauge transformation $[u,\cdot]$ and Spin(2) Lorentz rotation $T_\epsilon(\mathrm{Spin}(2)_{X_1,X_2})$ in the plane (X_1, X_2)

$$\begin{aligned} \delta_\epsilon\psi &= H\,, & \delta_\epsilon H &= [u,\psi]\,, \\ \delta_\epsilon\bar{u} &= \eta\,, & \delta_\epsilon\eta &= [u,\bar{u}]\,, \\ \delta_\epsilon X^i &= \chi^i\,, & \delta_\epsilon\chi^i &= [u, X^i] + \epsilon\varepsilon_{ij}X^j\,. \end{aligned} \tag{2.12}$$

The deformation method still works if we deal with Lorentz and gauge invariant expressions. After the deformation $\delta \to \delta_\epsilon$ the additional term $\epsilon[\bar{u}, X^i]\varepsilon_{ij}X^j$ is added to the action. By scaling of variables of integration it can be shown that the integral does not depend on ϵ. If we also require the limit $\epsilon \to 0$ not to be singular, then the value of the deformed integral will not change.

So, we continuously deform the original δ-exact action (2.11) to another one, more suitable for computation:

$$S_\epsilon^{\mathrm{def}} = \frac{1}{g}\delta_\epsilon(X_i\chi_j\varepsilon_{ij} + \bar{u}\psi) = \frac{1}{g}\Big[(\chi_i\chi_j + X_i[u,X_j])\ \ \varepsilon_{ij} - \epsilon X_iX_i + (\eta\psi + \bar{u}H)\Big]. \quad (2.13)$$

Then the trivial gaussian integration on $X^i, \chi^i, \bar{u}, \eta, H, \psi$ gives

$$\begin{aligned} I_b &= \frac{1}{\mathrm{Vol}(G/Z)}\int du\, dX^i d\chi^i\, dH d\psi\, d\bar{u} d\eta\ e^{1/g[(\chi_i\chi_j + X_i[u,X_j])\varepsilon_{ij} - \epsilon X_iX_i + (\eta\psi + \bar{u}H)]} \\ &= \frac{1}{\mathrm{Vol}(G/Z)}\int du\, \frac{1}{\det(ad(u)+\epsilon)}. \end{aligned} \quad (2.14)$$

Now we see, that the deformation $\delta \to \delta_\epsilon$ resolves the singularity $\det(ad(u)) = 0$[6] by introducing into the action the mass term for X_1, X_2. Then, using the gauge invariance one can reduce the integration over the whole Lie algebra $\mathfrak{g}$ to the integration over its r-dimensional Cartan subalgebra $\mathfrak{h}$

$$I_b^{\mathrm{MNS}}(G) = \frac{1}{\mathrm{Vol}(G/Z)}\int_{\mathfrak{g}} du\, \frac{1}{\det(ad(u)+\epsilon)} = \frac{1}{\#W}\int_{\mathfrak{h}} \frac{d^r u}{\mathrm{Vol}(T/Z)}\, \frac{\det'(ad(u))}{\det(ad(u)+\epsilon)}. \quad (2.15)$$

By $\#W$ we denote the order of the corresponding Weyl group and by $\#Z$ the order of the center of G. The factor $\det'(ad(u))$[7] in the numerator appears from the volume of the orbit of $u \in \mathfrak{h}$ obtained by a G-adjoint action.

Then we can explicitly rewrite the integral only in terms of roots $\{\alpha\}$ of the algebra $\mathfrak{g}$

$$I_b^{\mathrm{MNS}}(\mathfrak{g}) = \frac{\det\|\alpha^s_{ij}\|}{\#W}\int_{\mathcal{C}} du_1 \int_{\mathcal{C}} du_2 \cdots \int_{\mathcal{C}} du_r\, \frac{1}{(2\pi i\epsilon)^r}\prod_\alpha \frac{\alpha u}{\alpha u + \epsilon}, \quad (2.16)$$

(where $\{\alpha^s\}$ is the set of simple roots).

The performed deformation is valid if the domain of integration is compactified and the integrand is not singular on it. The compact domain of integration implies that the point $u = \infty$ should be included to it, and the reiterative integrals in (2.16) should be taken along the closed contour $\mathcal{C}$. The integrand in (2.16) has poles at the points $\alpha u + \epsilon = 0$ and $u = \infty$, and contour of integration should not pass throw them. Therefore, we shift the initial contours of integration for $u^i \in \overline{\mathbb{C}}$ from the point $u^i = \infty$. In terms of a complex plane $\mathbb{C}$ this means that the deformed contours of integrations for every u^i are now consist of the real axis $\mathbb{R} \subset \mathbb{C}$ and are closed by the infinite upper or lower arc.[8]

Despite the performed reduction of the initial integral I_b to the contour integral I_b^{MNS} might seem to be not rigorous enough, we do emphasize that it was strongly supported by the direct numerical (Monte-Carlo) evaluation of the integral I_b for all simple groups of rank $\leqslant 3$ in [12].

[6] $\det(ad(u)) = 0$ since the adjoint action of u vanishes on elements from Cartan subalgebra $\mathfrak{h}$.

[7] by $\det'(ad(u))$ we denote the determinant of $ad(u)$ acting on $\mathfrak{g} \setminus \mathfrak{h}$.

[8] It really does not matter whether we close contours by the upper or the lower arc, but it is necessarily to close contours in the same manner for all u^i.

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3. Evaluating of the contour integral $I_b^{\mathrm{MNS}}(\mathfrak{g})$

The contour integral (2.16) of a rational function can be computed by residues and it was done in the original work [10] for SU(N).

The crucial step in that computation was to represent a large product over the set of roots of A_{N-1} as a sum over permutations (it is the SU(N) Weyl group) of rather simple terms with a help of "the determinant formula" that was mentioned in the Introduction.

$$\frac{1}{\epsilon^r}\prod_{i\neq j}\frac{u_i-u_j}{u_i-u_j+\epsilon}=\sum_{\sigma\in\mathcal{S}_N}\prod_{i=1\ldots N}\frac{1}{u_i-u_{\sigma(i)}+\epsilon}\,. \tag{3.1}$$

After evaluation of contour integral by residues it can be seen, that only terms corresponding to the longest cycles (there are $(N-1)!$ such cycles) of permutations $\mathcal{S}_N$ remain. Each such term contributes to the sum $1/N^2$. Thus, the MNS result is

$$I_b^{\mathrm{MNS}}(\mathrm{SU}(N))=\frac{N}{N!}\frac{(N-1)!}{N^2}=\frac{1}{N^2}\,. \tag{3.2}$$

The additional factor N in the numerator is the order of the center Z_N of SU(N).

Our task is to evaluate the integral $I_b^{\mathrm{MNS}}(\mathfrak{g})$ for arbitrary simple Lie algebra. The direct extension of the MNS method for arbitrary simple Lie algebra would be to find an analogue of "the determinant formula" that represent product over all roots of an algebra $\mathfrak{g}$ in eq. (2.16) as a sum over its Weyl group of simpler terms,[9] suitable for a computation of the contour integral. Unfortunately, we have not managed to find an analogue that would be relevant in our case. For the exceptional groups the integral I_b^{MNS} can be explicitly evaluated, and it was done in [12] for G_2. We computed also I_b^{MNS} for F_4 and E_6 and so only E_7 and E_8 values are still unknown now.

As regards the other infinite classical series B_N,C_N,D_N we explicitly computed $I_b^{\mathrm{MNS}}(\mathfrak{g})$ for $N\leqslant 11$ and conjectured general formulas for every $N\in\mathbb{N}$. For all groups, except those that are isomorphic to the unitary one, our results do not coicide with the KS expression (1.16). However, we explicitly evaluated the KS sums (5.1) over Weyl groups for the classical series B_N,C_N,D_N and can provide a suggestion on how KS formula could be modified to give results agreeing with I_b^{MNS}.

4. Explicit results

To our best knowledge of the literature, it seems that the reliable results for the matrix integral I_b have been obtained before for the following groups

- SU(N),[10] $\forall N$ [10]
- SO($2N$), SO($2N+1$), Sp($2N$),[11] $N\leqslant 3$ [12]
- G_2 [12]

[9]These terms may be products over some subsets of the set of roots.
[10]A_N-series.
[11]D_N,B_N,C_N-series.

So, the entire infinite series B_N, C_N, D_N and the remaining exceptional groups (F_4, E_6, E_7, E_8) have to be explored to complete the story.

In addition to the previous results, in this work we explicitly computed the contour integral (2.16) for the following groups:[12]

- $\mathrm{Sp}(2N), \mathrm{SO}(2N), \mathrm{SO}(2N+1), \qquad N \leqslant 11$
- F_4
- E_6

Here is the table of our results for the series B_N, C_N, D_N:

N	$\mathrm{Sp}(2N)$ $[C_N]$	$\mathrm{SO}(2N+1)$ $[B_N]$	$\mathrm{SO}(2N)$ $[D_N]$
1	1/4	1/4	0
2	9/64	9/64	1/16
3	51/512	25/256	1/16
4	1275/16384	613/8192	117/2048
5	8415/131072	1989/32768	53/1024
6	115005/2097152	26791/524288	6175/131072
7	805035/16777216	92599/2097152	5661/131072
8	45886995/1073741824	5220675/134217728	1338019/33554432
9	331406075/8589934592	18671491/536870912	310819/8388608
10	4838528695/137438953472	270276175/8589934592	74352375/2147483648
11	35629165845/1099511627776	987486975/34359738368	69819475/2147483648

and for the exceptional groups[13]

G_2	$\frac{1}{12}\frac{151}{72}$
F_4	$\frac{1}{1152}\frac{493013}{3456}$
E_6	$\frac{1}{51840}\frac{286340}{81}$

The first lines of these tables $N \leqslant 3$ are in a perfect agreement with the previous results of [12] where similar integrals for all simple algebras with rank $\leqslant 3$ have been computed.

Our conjectured generic formulas for the classical infinite series B_N, C_N, D_N can be found in the eqs. (5.19), (5.14) and (5.18) below.

5. The deficit terms I_d and the conjecture for a general formula for I_b^{MNS} in the case of $\mathrm{SO}(2N+1), \mathrm{Sp}(2N+1), \mathrm{SO}(2N)$ groups

In [9] the following formula for $I_d(\mathfrak{g})$ was suggested.

$$I_d^{\mathrm{KS}}(\mathfrak{g}) = \frac{1}{\#W} \sum_{w \in W}{}' \frac{1}{\det(1-w)} \,. \tag{5.1}$$

[12] A special symbolic manipulation program was written to sum up a huge number of residues (its number grows faster then $N!$ and so we are limited by rather small N).

[13] The first fraction of these numbers is $1/\#W$.

As it was stated above this expression does not agree with the result obtained by the MNS deformation technique[14] for all simple groups except the unitary one (or isomorphic to it).[15]

In the case of the unitary group SU(N) the expression (5.1) can be easily computed. Indeed, in the sum over permutations $\mathcal{S}_N$ only the longest cycles contribute $\det(1-w)^{-1} = 1/N$, while the others have $\det(1-w) = 0$ and thus are thrown away. There are $(N-1)!$ longest cycles. So, the result is

$$I_d^{\mathrm{KS}}(\mathfrak{su}(N)) = \frac{1}{N!}\frac{(N-1)!}{N} = \frac{1}{N^2} \tag{5.2}$$

and it is correct.

Let us consider now the cases of the classical series B_N, C_N, D_N, where the KS formula (5.1) does not agree with the MNS contour integral (2.16) in known examples.

We have managed to explicitly compute the KS formula (5.1) as a function of rank N for all classical series (SO($2N+1$),Sp($2N$), SO($2N$)).

Firstly, let us consider C_N-case (Sp($2N$)). The C_N Weyl group W consists of permutations and changing signs of u_i.

$$W = (Z_2)^N \ltimes \mathcal{S}_N\,. \tag{5.3}$$

We can classify all elements of W in the following way. Every permutation can be broken into the direct product of l cycles c_{k_i} of lengths $k_1, k_2 \dots k_l$.

$$\det(1-w) = \prod_{i=1\dots l} \det(1-c_{k_i})\,. \tag{5.4}$$

Then, for every cycle c we note the following identity

$$\det(1-c) = 1 + \det(-c) \tag{5.5}$$

(Proof. Consider the determinant of the corresponding matrix $1-c$ as a tautological sum over all permutations of products $\prod_{i=1\dots k}(1-c)_{i,\sigma(i)}$. In this sum only two terms corresponding to the product of 1 on the main diagonal and elements of matrix c contribute).

The determinant of cycle can be easily computed:

$$\det(-c_{k_i}) = (-1)^{k_i}(-1)^{k_i+1}(-1)^{\#-} = (-1)^{1+\#-}\,. \tag{5.6}$$

Thus, $\det(1-c)$ is equal to 2 or to 0 in equal number of cases.[16]

A number of ways of choosing long cycles $k_1, \dots k_l$ is $\frac{N!}{k_1!\dots k_l!}$. Each cycle k_i can be realized in $2^{k_i}(k_i-1)!$ ways and only a half of them contribute to the corresponding term an additional factor $1/2$ each.

[14] but that results are strongly supported by numerical Monte-Carlo evaluation in [12].

[15] The fractional parts of I_b and I_d should coincide regardless of the value of ind_w (since it is integer), but they do not. In the following, we will conjecture that $\mathrm{ind}_w = 0$ since in all known cases $I_b < 1$, $I_d > 0$ and $0 \leqslant \mathrm{ind}_w = I_b - I_d$.

[16] It is non zero when the number of minus signs is odd. It means that this element of the Weyl group considered as a permutation of $2k_i$ elements $\{\pm u_i\}$ is a cycle of length $2k_i$ and cannot be broken into two disjoint cycles of length k_i as it would when $\#-$ is even.

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So, we obtain

$$\sum_{w\in W_{\mathrm{Sp}(2N)}}^{\prime} \frac{1}{\det(1-w)} =$$
$$= \sum_{l=1}^{N} \frac{1}{l!} \sum_{k_1+k_2+\cdots+k_l=N} \frac{N!}{k_1!k_2!\dots k_l!}(k_1-1)!(k_2-1)!\cdots(k_l-1)!\frac{2^{k_1+k_2+\cdots+k_l}}{2^l 2^l}$$
$$= 2^N \sum_{l=1}^{N} \frac{2^{-2l}}{l!} \sum_{k_1+k_2+\cdots+k_l=N} \frac{N!}{k_1 k_2 \dots k_l}\,. \tag{5.7}$$

Then, using the identity

$$\frac{1}{x_1 x_2 \cdots x_N} = \sum_{\sigma\in S_N} \frac{1}{x_{\sigma(1)}\left(x_{\sigma(1)}+x_{\sigma(2)}\right)\cdots\left(x_{\sigma(1)}+\cdots+x_{\sigma(N)}\right)} \tag{5.8}$$

and making change of variables $k_i \to \sum_{j=1}^{i} k_j$, we obtain the following expression for the number of permutations consisting of l disjoint cycles (it is an absolute value of the Stirling number $S(N,l)$ of the first kind)

$$F_N^l \equiv \frac{1}{l!} \sum_{k_1+k_2+\cdots+k_l=N} \frac{N!}{k_1 k_2 \dots k_l} =$$
$$= \sum_{k_1+k_2+\cdots+k_l=N} \frac{N!}{k_1(k_1+k_2)\cdots(k_1+\cdots+k_l)}$$
$$= \sum_{1\leqslant k_1<k_2<\cdots<k_l=N} \frac{N!}{k_1 k_2 \dots k_l}$$
$$= \sum_{1\leqslant k_1<k_2<\cdots<k_{N-l}\leqslant N-1} k_1 k_2 \dots k_{N-l}\,. \tag{5.9}$$

Finally, we obtain

$$\sum_{w\in W_{\mathrm{Sp}(2N)}}^{\prime} \frac{1}{\det(1-w)} = \sum_{l=1}^{N} 2^{N-2l} F_N^l = 2^N \sum_{l=1}^{N} \frac{1}{2^{2l}} \sum_{1\leqslant k_1<k_2<\cdots<k_{N-l}\leqslant N-1} k_1 k_2 \dots k_{N-l}$$
$$= 2^N \prod_{k=0}^{N-1}\left(k+\frac{1}{4}\right) = 2^{-N}\prod_{k=0}^{N-1}(4k+1)\,. \tag{5.10}$$

So, the $\mathrm{Sp}(2N)$ KS formula (5.1) is reduced to the following function of N

$$I_d^{\mathrm{KS}}(\mathfrak{sp}(2N)) = \frac{1}{2^{2N}N!}\prod_{k=0}^{N-1}(4k+1)\,. \tag{5.11}$$

This formula provides the correct value $1/4$ for $N=1$ (in this case $\mathrm{Sp}(2)$ is isomorphic to the unitary group $\mathrm{SU}(2)$), but does not coincide with the contour integral I_b^{MNS} for all other $N>1$. For $N=2$ we have

$$\#W(C_2)\, I_d^{\mathrm{KS}}(\mathfrak{sp}(4)) = 1 + 1/4 = \frac{5}{4}\,, \tag{5.12}$$

while

$$\#W(C_2)\, I_b^{\mathrm{MNS}}(\mathfrak{sp}(4)) = \frac{9}{8}\,. \tag{5.13}$$

However, we note, that the results for the contour integral $I_b^{\mathrm{MNS}}(\mathfrak{sp}(2N))$ obtained by the explicit calculation for $N \leqslant 11$ are perfectly described by the following function

$$I_b^{\mathrm{MNS}}(\mathfrak{sp}(2N)) = \frac{1}{2^{3N-1}N!}\prod_{k=0}^{N-1}(8k+1) \tag{5.14}$$

which is different, but still rather similar to (5.11).

If one converts this expression into a form similar to the (5.10), it can be seen that to make KS formula (5.1) agree with I_b^{MNS} in the $\mathrm{Sp}(2N)$ case it is sufficient to multiply terms in the sum corresponding to l disjoint permutation cycles by an additional factor $1/2^{l-1}$. We can interpret this correction in the sense that every additional disjoint permutation cycle contributes another factor $1/2$ into the products in the sum (5.10) $\sum_{l=1}^{N} 2^{N-2l} F_N^l \to \sum_{l=1}^{N} 2^{N-2l} 2^{1-l} F_N^l$.

It is also a curios fact that in both cases of $\mathrm{SU}(N)$ and $\mathrm{Sp}(2N)$ groups[17] the following expression for I_d agrees with I_b^{MNS}

$$I_d(G) = \frac{\#Z}{\#W}\sum_{w\in W}{}' \frac{1}{\det(1-w)^2}\,, \qquad G \in \{\mathrm{SU}(N), \mathrm{Sp}(2N)\}\,. \tag{5.15}$$

We have also managed to explicitly compute the KS formula (5.1) for the D_N Weyl group. The D_N Weyl group can be realized in the same manner as the B_N Weyl group with an additional requirement that a number of changed signs $u_i \to -u_i$ should be even. In order to calculate (5.1) we now need to sum up only over permutations that consist of cycles with an even number of minus signs. That means that the terms with an odd number of cycles in the permutation should be projected out. So, we obtain

$$\begin{aligned}\sum_{w\in W_{\mathrm{SO}(2N)}}{}' \frac{1}{\det(1-w)} &= \sum_{l=1}^{N} 2^{N-2l} F_N^l \frac{(1+(-1)^l)}{2} \\ &= 2^{-N-1}\left(\prod_{k=0}^{N-1}(4k+1) + \prod_{k=0}^{N-1}(4k-1)\right)\end{aligned} \tag{5.16}$$

and finally

$$I_d^{\mathrm{KS}}(\mathfrak{so}(2N)) = \frac{2}{2^{N-1}N!}2^{-N-1}\left(\prod_{k=0}^{N-1}(4k+1) + \prod_{k=0}^{N-1}(4k-1)\right). \tag{5.17}$$

Again, KS formula I_d^{KS} (5.1) does not agree with the MNS contour integral I_b^{MNS}. But we observe that the result for the MNS contour integral (2.16) in the explicitly computed $\mathrm{SO}(2N+1)$ and $\mathrm{SO}(2N)$ ($N \leqslant 11$) cases can be obtained by multiplying the terms in the

[17] But not for every simple Lie group, of course!.

sum (5.10) and in the middle part of (5.16), respectively, by the same factors from the sequence $b_l = \{1, 1, -2, -2, 76, 76, \ldots\}$.

$$I_b^{\mathrm{MNS}}(\mathfrak{so}(2N+1)) = \frac{1}{\#W} \sum_{l=1}^{N} 2^{N-2l} 2^{1-l} F_N^l \; b_l \,, \tag{5.18}$$

$$I_b^{\mathrm{MNS}}(\mathfrak{so}(2N)) = \frac{2}{\#W} \sum_{\text{even } l=2}^{N} 2^{N-2l} 2^{1-l} F_N^l \; b_l \,. \tag{5.19}$$

Obviously, given arbitrary sequence of numbers X_N[18] and sequences $\{f_N^l \neq 0, \; l = \overline{1 \ldots N}\}$[19] we can always linearly expand X_N over the set f_N^l

$$\sum_{l=1}^{N} f_N^l b_l = X_N \,. \tag{5.20}$$

The coefficients b_l can be recurrently found:

$$\begin{aligned} b_1 &= \frac{X_1}{f_1} \,, \\ &\vdots \\ b_N &= \frac{\left(X_N - \sum_{l=1}^{N-1} f_l b_l \right)}{f_N} . \end{aligned} \tag{5.21}$$

In this way we can unambiguously get the two sequences $\{b(\mathrm{SO}(2N))\}$ and $\{b((\mathrm{SO}(2N+1))\}$. Then, we note that up to $N \leqslant 11$

1. These two sequences

$$\{b_{2k}(\mathrm{SO}(2N))\} \,,$$
$$\{b_{2k}(\mathrm{SO}(2N+1))\}$$

have coincided with each other in spite of the fact that the algebras, root systems and the contour integral expressions 2.16 are different!

$$b_{2k}(\mathrm{SO}(2N)) \;= b_{2k}(\mathrm{SO}(2N+1)) \,, \qquad \forall k \,. \tag{5.22}$$

2. In the $\mathrm{SO}(2N+1)$ case when in the sum also terms with odd $l = 2k+1$ are present, the coefficients b_{2k+1} are equal to the preceding ones b_{2k}.

$$b_{2k+1} = b_{2k} \,, \qquad \forall k \,. \tag{5.23}$$

3. The signs of pairs $(b_{2k} = b_{2k+1})$ are interchanged.

$$\operatorname{sign} b_{2k} = (-1)^{k+1} \,. \tag{5.24}$$

[18] In the following we will substitute X_N by $I_b^{\mathrm{MNS}}(\mathfrak{so}(2N+1))$ or $I_b^{\mathrm{MNS}}(\mathfrak{so}(2N))$.
[19] by f_N^l we mean $\frac{\#Z}{\#W} 2^{N-2l} 2^{1-l} F_N^l$.

Let us represent the sequence b_{2k} in the following way:

$$b_{2k} = (-1)^{k+1} 2^k \beta_k \,. \tag{5.25}$$

Then $\beta_k = \{1, 1, 19, 559, 29161, \ldots\}$.

4. The most non-trivial observation and indeed a conjecture about a value for the MNS contour integral $\forall N \in \mathbb{N}$ in the $\mathrm{SO}(2N)$ and $\mathrm{SO}(2N+1)$ case is the following. The coefficients β_k are expressed as numerators in the Taylor expansion of the generating function $\sqrt{\cos(x)}$.

$$\sqrt{\cos x} = 1 - \sum_{k=0}^{\infty} \frac{\beta_k x^{2k}}{2^k (2k)!} \,. \tag{5.26}$$

We conjecture that these observable properties of b_{2k} and b_{2k+1} for $N \leqslant 11$ are extended for arbitrary N.

6. Conclusion

In this work the MNS contour integral $I_b^{\mathrm{MNS}}(\mathfrak{g})$ has been explicitly evaluated in the case of classical $\mathrm{SO}(2N+1), \mathrm{Sp}(2N), \mathrm{SO}(2N)$ groups for $N \leqslant 11$ and also for the exceptional groups G_2, F_4, E_6. The conjectures about values of $I_b^{\mathrm{MNS}}(G)$ in the case of classical infinite series $\mathrm{SO}(2N+1), \mathrm{Sp}(2N), \mathrm{SO}(2N)$ have been suggested $\forall N \in \mathbb{N}$. In addition, KS formula (5.1) for $I_d^{\mathrm{KS}}(\mathfrak{g})$ has been explicitly computed as a function of rank N in the $\mathrm{SO}(2N+1), \mathrm{Sp}(2N), \mathrm{SO}(2N)$ cases. The results disagreed with the values of I_b^{MNS} but the computation helped us to conjecture the value of $I_b^{\mathrm{MNS}}(\mathfrak{g})$ in the case of classical infinite series $\mathrm{SO}(2N+1), \mathrm{Sp}(2N), \mathrm{SO}(2N)$.

For $\mathrm{Sp}(2N)$ series it is simple:

$$I_b^{\mathrm{MNS}}(\mathfrak{sp}(2N)) = \frac{1}{2^{3N-1} N!} \prod_{k=0}^{N-1} (8k+1) \,. \tag{6.1}$$

For $\mathrm{SO}(2N+1)$ and $\mathrm{SO}(2N)$ series the result is build with the help of the generating function $\sqrt{\cos(x)}$ in the following way:

1) Define β_k as

$$\sqrt{\cos x} \equiv 1 - \sum_{k=0}^{\infty} \frac{\beta_k x^{2k}}{2^k (2k)!} \,. \tag{6.2}$$

2) Define b_{2k} and b_{2k+1}

$$b_{2k} \equiv (-1)^{k+1} 2^k \beta_k \,. \tag{6.3}$$

3) Define $F_N^l = S(N, l)$ as Stirling number of the first kind

$$F_N^l \equiv \sum_{1 \leqslant k_1 < k_2 < \cdots < k_{N-l} \leqslant N-1} k_1 k_2 \ldots k_{N-l} \,. \tag{6.4}$$

Then the result is

$$I_b^{\rm MNS}({\rm SO}(2N+1)) = \frac{1}{2^N N!}\sum_{l=1}^{N} 2^{N+1-3l}F_N^l b_k\,, \tag{6.5}$$

$$I_b^{\rm MNS}({\rm SO}(2N)) = \frac{2}{2^{N-1}N!}\sum_{\text{even } l=2}^{N} 2^{N+1-3l}F_N^l b_k\,. \tag{6.6}$$

And here are the explicit expressions for $I_d^{\rm KS}(\mathfrak{g})$ for the orthogonal and symplectic groups

$$I_d^{\rm KS}(\mathfrak{sp}(2N)) = I_d^{\rm KS}({\rm SO}(2N+1) = \frac{1}{2^{2N}N!}\prod_{k=0}^{N-1}(4k+1)\,, \tag{6.7}$$

$$I_d^{\rm KS}(\mathfrak{so}(2N)) = \frac{2}{2^{N-1}N!}2^{-N-1}\left(\prod_{k=0}^{N-1}(4k+1)+\prod_{k=0}^{N-1}(4k-1)\right). \tag{6.8}$$

7. Outlook or what remains to do?

1. To prove our conjectures about the values of $I_b^{\rm MNS}$ for the classical series B_N, C_N, D_N for arbitrary N. We suggest that this part is technical since one needs to prove pure combinatorial identity.

2. To understand on an algebraic level how the MNS contour integral is connected with the KS (5.1) formula. (We observed that it is sufficient to replace

$$\det(1-w) \longrightarrow \det(1-w)^2$$

and multiply terms in the sum by some additional integer tuning factors (like b_k). This procedure works for G_2 group too (see appendix). But what is hidden behind this manipulation?) And what exactly is wrong about the effective free hamiltonian method at large X_i in super Yang-Mills quantum mechanics?

3. Perhaps, it possible to find such a deformation of the initial integral I_b that will be suitable for using some generalization of "the determinant formula" (that one also needs to find) and directly deduce our conjectures.

4. To understand in a more direct way how the generating function $\sqrt{\cos(x)}$ is connected with ${\rm SO}(2N), {\rm SO}(2N+1)$ $D=0$ $\mathcal{N}=4$ super Yang-Mills matrix model.

5. To accurately perform a reduction of the matrix integral I_b to the contour integral $I_b^{\rm MNS}$ and obtain a rigorous proof of the way that was used to deform the integration contours.

6. To obtain the values of the matrix integral I_b for arbitrary simple group in the case of dimensional reduction to 0 dimension of $D=6$ and $D=10$ super Yang-Mills. In the ${\rm SU}(N)$ $D=10$ case the result was computed in the same paper [10]

$$I_b = \sum_{d\,|N \bmod\ d=0}\frac{1}{d^2} \tag{7.1}$$

but what about other simple groups?

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A. Explicit expressions for contour integrals

Here we provide explicit expressions for the contour integrals

$$I_b^{\mathrm{MNS}}(G) = \frac{1}{\mathrm{Vol}(T/Z)\#W} \int d^r u \, \frac{1}{(i\epsilon)^r} \prod_\alpha \frac{\alpha u}{\alpha u + \epsilon} \tag{A.1}$$

for various simple Lie algebras expressed through their root systems. The factor $1/\mathrm{Vol}(T/Z)$ in front of the integral can be rewritten as $k/(2\pi)^r$, where k is the volume of the cell spanned by the set of simple roots α_i^s in the weight space:

$$k = \det \|\alpha_{ij}^s\| \tag{A.2}$$

and

$$I_b^{\mathrm{MNS}}(G) = \frac{k}{\#W} \int d^r u \, \frac{1}{(2\pi i\epsilon)^r} \prod_\alpha \frac{\alpha u}{\alpha u + \epsilon} \, . \tag{A.3}$$

A.1 $\mathrm{SO}(2N+1)$ or B_N

The root system is the set

$$\{\alpha u\} = \{\pm u_i \pm u_j \, , \qquad i < j\} \cup \{\pm u_i\} \, . \tag{A.4}$$

The Weyl group consists of permutations and changing signs of u_i.

$$W = (Z_2)^N \ltimes \mathcal{S}_N \, . \tag{A.5}$$

And

$$\#W = 2^N N! \, , \qquad k = 1 \, , \tag{A.6}$$

$$\begin{aligned} I_b^{\mathrm{MNS}}(\mathrm{SO}(2N+1)) &= \frac{1}{2^N N!} \frac{1}{(2\pi i \epsilon)^N} \times \\ &\times \int d^N u \prod_{i<j} \frac{(u_i+u_j)^2 (u_i-u_j)^2}{((u_i+u_j)^2-\epsilon^2)((u_i-u_j)^2-\epsilon^2)} \prod_i \frac{u_i^2}{(u_i^2-\epsilon^2)} \, . \end{aligned} \tag{A.7}$$

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A.2 $\mathrm{Sp}(2N)$ or C_N

The root system is the set

$$\{\alpha u\} = \{\pm u_i \pm u_j\,, \qquad i<j\} \cup \{\pm 2u_i\}\,. \tag{A.8}$$

The Weyl group is the same as for $\mathrm{SO}(2N+1)$. It consists of permutations and changing signs of u_i.

$$W = (Z_2)^N \ltimes \mathcal{S}_N\,. \tag{A.9}$$

And

$$\#W = 2^N N!\,, \qquad k=2\,, \tag{A.10}$$

$$I_b^{\mathrm{MNS}}(\mathrm{Sp}(2N)) = \frac{2}{2^N N!}\frac{1}{(2\pi i\epsilon)^N}\times \\ \times \int d^N u \prod_{i<j}\frac{(u_i+u_j)^2(u_i-u_j)^2}{((u_i+u_j)^2-\epsilon^2)((u_i-u_j)^2-\epsilon^2)}\prod_i \frac{4u_i^2}{(4u_i^2-\epsilon^2)}\,. \tag{A.11}$$

A.3 $\mathrm{SO}(2N)$ or D_N

The root system is the set

$$\{\alpha u\} = \{\pm u_i \pm u_j\,, \qquad i<j\}\,. \tag{A.12}$$

The Weyl group is the subgroup of the B_N Weyl group. It consists of permutations and changing *even number* of signs of u_i.

$$W = Z_2^{N-1} \ltimes \mathcal{S}_N\,. \tag{A.13}$$

And

$$\#W = 2^{N-1} N!\,, \qquad k=2\,, \tag{A.14}$$

$$I_b^{\mathrm{MNS}}(\mathrm{SO}(2N)) = \frac{2}{2^{N-1} N!}\frac{1}{(2\pi i\epsilon)^N}\times \\ \times \int d^N u \prod_{i<j}\frac{(u_i+u_j)^2(u_i-u_j)^2}{((u_i+u_j)^2-\epsilon^2)((u_i-u_j)^2-\epsilon^2)}\,. \tag{A.15}$$

A.4 G_2

The root system is the set

$$\{\alpha u\} = \left\{\frac{1}{\sqrt{3}}(u_i-u_j)\,, \qquad i\neq j\right\} \cup \left\{\pm\frac{1}{\sqrt{3}}(u_i+u_j-2u_k)\,, \qquad i\neq j\neq k\right\}, \\ i,j,k \in \{1,2,3\}\,. \tag{A.16}$$

Note, that the root lattice is restricted to the plane $\alpha_1+\alpha_2+\alpha_3=0$ and thus it is two-dimensional.

The Weyl group consists of permutations and changing simultaneously all signs of u_i.

$$W = Z_2 \ltimes \mathcal{S}_3\,. \tag{A.17}$$

And

$$\#W = 12\,, \qquad k = 1\,, \tag{A.18}$$

$$I_b^{\mathrm{MNS}}(G_2) = \frac{1}{12}\int du_1 du_2 \frac{1}{(2\pi i\epsilon)^2} \prod_{\alpha\in\Delta(G_2)} \frac{\alpha u}{\alpha u+\epsilon}\Bigg|_{u_3=-u_2-u_1} . \tag{A.19}$$

A.5 F_4

The root system is the set

$$\{\alpha u\} = \{\pm u_i\} \cup \{\pm u_i \pm u_j\,, i\neq j\} \cup \left\{\frac{1}{2}(\pm u_1 \pm u_2 \pm u_3 \pm u_4)\right\}. \tag{A.20}$$

The Weyl group is a group of automorphisms of the lattice $Q(D_4)$.

And

$$\#W = 24 * 48 = 1152\,, \tag{A.21}$$

$$k = \frac{1}{2}\,, \tag{A.22}$$

$$I_b = \frac{1}{2\cdot 1152}\int d^4u \frac{1}{(2\pi i\epsilon)^r} \prod_{\alpha\in\Delta(F_4)} \frac{\alpha u}{\alpha u+\epsilon}\,. \tag{A.23}$$

A.6 E_6

The root system is the set

$$\{\alpha u\} = \left\{\pm\sqrt{2}u_7\right\} \cup \{u_i - u_j, i, j \leqslant 6\} \cup \left\{\frac{1}{2}\left(\sum_{i=1}^{6}\epsilon_i u_i \pm \sqrt{2}u_7\right), \epsilon_i = \pm 1\,, \sum_{i=1}^{6}\epsilon_i = 0\right\} \tag{A.24}$$

and the root lattice is restricted to the plane $\{a| \sum_{i=1}^6 a_i = 0\}$.

The Weyl group is a group of automorphisms of the lattice $Q(E_6)$.

And

$$\#W = 6! * 72\,, \tag{A.25}$$

$$k = 3\sqrt{2}\,, \tag{A.26}$$

$$I_b^{\mathrm{MNS}}(E_6) = \frac{3\sqrt{2}}{6! * 72}\int du_2 \ldots du_7 \frac{1}{(2\pi i\epsilon)^r} \prod_{\alpha\in\Delta(E_6)} \frac{\alpha u}{\alpha u+\epsilon}\Bigg|_{u_1=-(u_2+\cdots+u_6)} . \tag{A.27}$$

B. Comparison of I_d^{KS} and I_b^{MNS} for G_2 group

In the G_2 case the valid result 151/864 obtained by evaluating contour integral can be expressed in the manner similar to (5.15) and (5.18).

The G_2 Weyl group contains 12 elements $w \in W$. The following ones have non-zero $\det(1-w)$: 2 terms with $\det(1-w) = 1$, 2 terms with $\det(1-w) = 3$ and 1 term with $\det(1-w) = 4$. According to [9] we should get

$$I_d^{\mathrm{KS}}(G_2) = \frac{1}{12}\left(2\cdot 1 + 2\cdot\frac{1}{3} + \frac{1}{4}\right) = \frac{1}{12}\frac{35}{12} \tag{B.1}$$

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but $I_b^{\rm MNS}$ seems to be obtained as

$$I_b^{\rm MNS}(G_2) = \frac{1}{12}\left(2\cdot(1)^2 + 2\cdot\left(\frac{1}{3}\right)^2 + (-2)\cdot\left(\frac{1}{4}\right)^2\right) = \frac{1}{12}\frac{151}{72}\,. \tag{B.2}$$

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